# Joint Separation of Geometric Clusters and the Extreme Irregularities of Regular Polyhedra [*]

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### Abstract

We propose a natural scheme to measure the joint separation of a cluster of objects in general geometric settings. In particular, here the measure is developed for finite sets of planes in $\mathbb{R}^3$ in terms of extreme configurations of vectors on the planes of a given set. We prove geometric and graph-theoretic results about extreme configurations on arbitrary finite plane sets. We then specialize to the planes bounding a regular polyhedron in order to exploit the symmetries. However, even then results are non-trivial and surprising – extreme configurations on regular polyhedra may turn out to be highly irregular.

*Keywords:* Computational geometry, cluster, configuration, Platonic solid, polyhedron, regular polyhedron, separation.

## 1 Introduction

In [3] the question arose that if three vectors are arbitrarily chosen, lying one each on the three co-ordinate planes, what is the largest angle $\theta$ such that, *in all instances*, at least two of the vectors have an angle of at least $\theta$ between them. The answer of $\pi/3$ is not hard to see given the symmetries of the co-ordinate planes: see Figure 1 for a so-called extreme configuration where the vectors $OA$, $OB$ and $OC$ lie on the $yz-$, $xz$, and $xy-$ co-ordinate planes, making angles of $\pi/4$ with the positive directions of the axes of their respective planes, and angles of $\pi/3$ with each other.

It is natural to extend the question to arbitrary finite sets of planes. This leads to a measure of joint separation of such sets in terms of certain extreme configurations of vectors on the planes belonging to a given set. The measure, in fact, generalizes in a natural manner to clusters of objects in various geometric settings. As far as we are aware, such a measure of joint separation has not been investigated before.

We first define the measure for sets of planes in real 3-space $\mathbb{R}^3$.

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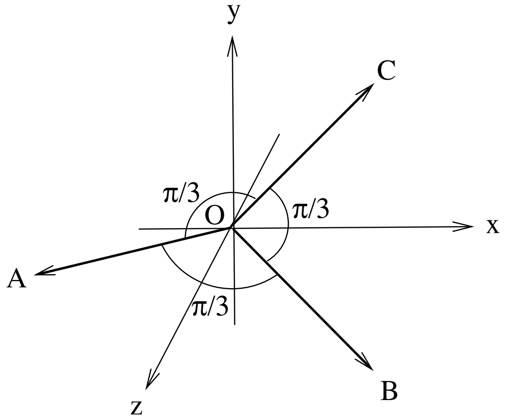


Figure 1: An extreme configuration on the three co-ordinate planes: $OA$, $OB$ and $OC$ lie on the $yz-$, $xz$, and $xy-$ co-ordinate planes, making angles of $\pi/4$ with the positive directions of the axes of their respective planes, and angles of $\pi/3$ with each other.

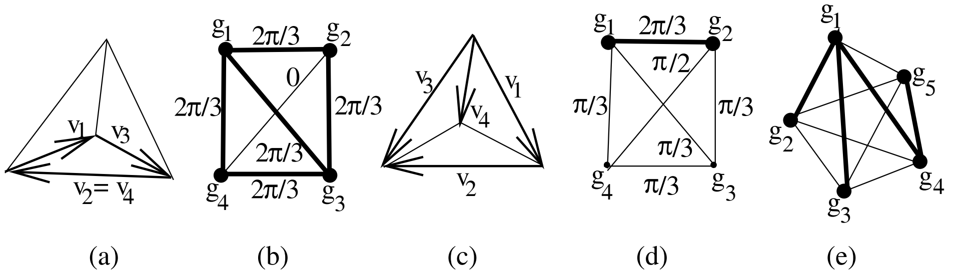


Figure 2: (a) and (c): Configurations on the four face planes of a regular tetrahedron; (b) and (d): Graphs of (a) and (c) – the subgraph of maximal edges is bold; (e) A graph of some configuration, weights not shown, where the subgraph (bold) of maximal edges is acyclic.

**Definitions.** Let $F$ be a finite set of planes of cardinality $n \geq 2$ in $\mathbb{R}^3$. A *spanning configuration of vectors*, or, simply, *configuration*, $C$ on $F$ is a set of $n$ non-null vectors, lying one each on a plane in $F$; in other words, there is one-to-one correspondence between $C$ and $F$ so that each vector in $C$ lies on its corresponding plane in $F$. We treat vectors as equivalence classes of arrows in $\mathbb{R}^3$ so parallel translates are identical. Moreover, both $F$ and $C$ are allowed to contain duplicates. See Figures 2(a) and (c).

The *angle* between a pair of non-null vectors $u$ and $v$ is $\angle uv = \cos^{-1} \frac{u \cdot v}{|u||v|}$, where $0 \leq \angle uv \leq \pi$.

Given a configuration $C$ on $F$, the *maximal angle* of the configuration is the maximum amongst angles between pairs of vectors from $C$, denote it by $max(C)$. E.g., both in Figures 2(a) and (c), it is $2\pi/3$. *Remark*: In determining $max(C)$ the length of each vector is immaterial – particularly, any vector in $C$ may be replaced by a parallel one (i.e., a positive multiple) – it is directions that matter. The maximal angle is evidently the diameter of $C$ using an appropriate angular metric.

An *extreme configuration* on $F$ is a configuration $C'$ such that

$$max(C') = \inf\{max(C) : C \text{ is a configuration on } F\}.$$

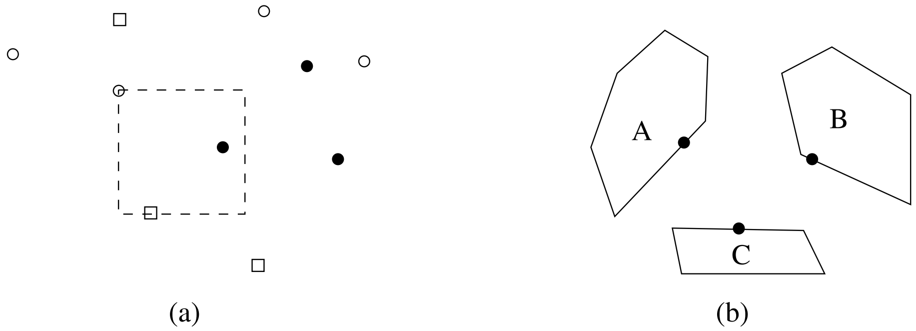


Figure 3: (a) Points of 3 colors on a plane with a set of 3 points of all different colors contained in the smallest square (equivalently, having the smallest $L_\infty$ diameter) shown; (b) 3 regions $A$, $B$ and $C$ with a set of points chosen from each such that the $L_2$ diameter is minimum.

Straightforward compactness and continuity arguments show that extreme configurations exist as $F$ is finite. If $C'$ is an extreme configuration on $F$, call $max(C')$ the *minmax angle* of $F$, denote it by $minmax(F)$.

Therefore, *a defining property of $minmax(F)$ is that it is the largest angle $\theta$ such that, if $n$ vectors are arbitrarily chosen, one each on a plane of $F$, then at least two of the vectors have an angle of at least $\theta$ between them.* Consequently, we say that $minmax(F)$ is a measure of *joint separation* of $F$.

The measure $minmax(F)$ of the joint separation of a finite set of planes $F$ in $\mathbb{R}^3$ generalizes naturally. A direct generalization of the geometric setting described above is to finite sets $F$ of affine subspaces of dimension at least one of some real space $\mathbb{R}^m$, where the definition of $minmax(F)$ remains unchanged. However, we describe next a more general setting for a *minmax-type* measure of joint separation.

**Definitions.** Let $\mathbb{X} = \{X_i : 1 \leq i \leq n\}$ be a collection of non-null subsets of a metric space $M$ with distance measure $d$. A spanning configuration $C$ on $\mathbb{X}$ is a sequence $x_1, x_2, \ldots, x_n$ of elements of $M$ such that $x_i \in X_i$, $1 \leq i \leq n$. The diameter of $C$, denote it $max(C) = \sup\{d(x_i, x_j) : 1 \leq i, j \leq n\}$. Define the *joint separation* of $\mathbb{X}$ by

$$minmax(\mathbb{X}) = \inf\{max(C) : C \text{ is a spanning configuration on } S\}$$

We describe a couple of practical scenarios to apply joint separation.

Consider $n$ points in real $m$-space $\mathbb{R}^m$, each having one of $k$ fixed colors. In other words, we have a cluster of $k$ finite subsets of $\mathbb{R}^m$. Problem: given an $L_p$ metric on $\mathbb{R}^m$, find a subset of $\mathbb{R}^m$ that contains one point of each color (i.e., a spanning configuration), of minimum diameter. See Figure 3(a). If one thinks of $k$ populations drawn from an $m$-attribute database, we are measuring a joint separation of the different populations. The $L_2$ and $L_\infty$ metrics seem most natural in the databases context.

Aggarwal et al [1] give efficient algorithms for the following related problems by applying higher-order Voronoi diagrams: given a set $S$ of $n$ points on a plane and a fixed $k$, find a subset of $S$ of size $k$ with minimum $L_2$ (or, $L_\infty$) diameter. It is not clear at present if their techniques generalize to the multi-colored setting.

Consider $k$ regions on the plane with the $L_2$ metric. Problem: find a set of points, one from each region, of minimum diameter. See Figure 3(b). Clearly, the measure of joint separation of the regions is relevant to, say, transmission facility location. We are not aware of efficient algorithms related to this problem.

Another simple but more theoretical example, motivated by the definition for planes: let $S$ be a set of oriented polygonal arcs $s_1, s_2, \ldots, s_n$, each associated with the set $X_i$ of oriented segments, normalized to unit length vectors, comprising it. Use the metric of angles between such vectors to define $minmax(S)$, similarly as for planes. Question: if the arcs in $S$ are all, say, known to be convex, is there an efficient way to compute $minmax(S)$?

The general notion of joint separation seems natural enough that one expects applications in various domains.

The rest of the paper is organized as follows. In Section 2 we prove geometric and graph-theoretic results about extreme configurations of vectors on finite sets of planes in $\mathbb{R}^3$. As computing $minmax(F)$ seems difficult for arbitrary set of planes $F$ in $\mathbb{R}^3$, we specialize in Section 3 to the planes bounding a regular polyhedron in order to exploit the symmetries. Even then results are non-trivial and surprising – extreme configurations on regular polyhedra may turn out to be highly irregular. We conclude in Section 4.

## 2 Planes in Real 3-Space

In this section we shall prove generally applicable geometric and graph-theoretic results about extreme configurations of vectors on finite sets of planes in $\mathbb{R}^3$, before specializing to the bounding planes of regular polyhedra in the next. We begin with a few definitions.

**Definitions.** If the startpoint of a *unit* vector $u$ is located at the origin then the endpoint of $u$ lies on the unit sphere $\mathbb{S}^2$. Since $u$ is uniquely identified by this endpoint we shall not distinguish between unit vectors and points on $\mathbb{S}^2$. A *unit vector trajectory*, or, simply, *trajectory*, is a *regular* $C^\infty$ curve (see [5] for definitions) $c : [a, b] \to \mathbb{S}^2$. The trajectory starts at $c(a)$ and ends at $c(b)$. If $p$ is a plane such that $c(t) \in p, \forall t \in [a, b]$, then the trajectory is said to lie on $p$. See Figure 4(a).

A *perturbation* of a set of unit vectors $\{v_i\}$ is set of trajectories $\{c_i\}$, such that $c_i$ starts at $v_i$ for each $i$. *Remark*: In the following, vectors in a configuration are often required to be of unit length to avoid technicalities and to allow application of perturbations in the manner defined.

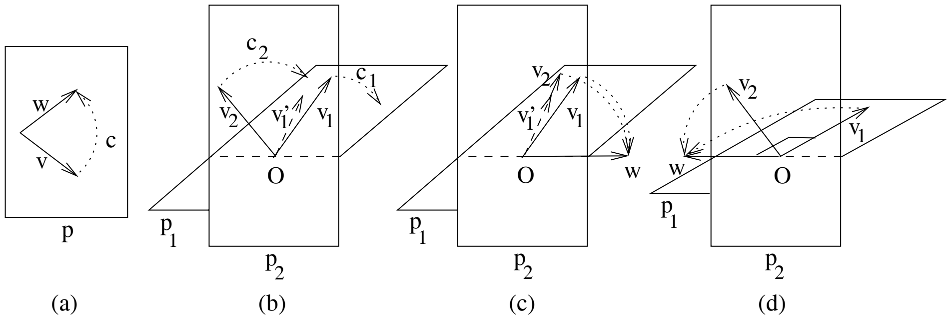
Informally, the first part of the following lemma says that two non-null vectors $v_1$ and $v_2$ lying on planes $p_1$ and $p_2$, respectively, such that the angle between $v_1$ and $v_2$ is greater than zero, can be rotated simultaneously a small amount to decrease that angle; the second part says that, under certain conditions, even if a rotation of one of the vectors is *given*, the other can be rotated to again decrease the angle between them.

**Lemma 1** *Suppose that $v_1$ and $v_2$ are unit vectors lying on planes $p_1$ and $p_2$, respectively, such that the angle between $v_1$ and $v_2$ is $\theta > 0$.*

*Then, there exist trajectories $c_i : [0, \epsilon_i] \to \mathbb{S}^2$, starting at $v_i$ and lying on $p_i$, for $1 \leq i \leq 2$, such that the angle between $c_1(t)$ and $c_2(t)$ is strictly less than $\theta$ for $0 < t \leq \delta$, for some positive $\delta \leq \min \epsilon_i$.*

*Further, if the projection $v_1'$ of $v_1$ on to $p_2$ is neither null nor parallel to $v_2$, then, given an arbitrary trajectory $c_1 : [0, \epsilon_1] \to \mathbb{S}^2$, starting at $v_1$ and lying on $p_1$, there exists a trajectory*

$c_2 : [0, \epsilon_2] \to \mathbb{S}^2$, *starting at* $v_2$ *and lying on* $p_2$, *such that the angle between* $c_1(t)$ *and* $c_2(t)$ *is strictly less than* $\theta$ *for* $0 < t \leq \delta$, *for some positive* $\delta \leq \min \epsilon_i$.



(a) (b) (c) (d)

Figure 4: (a): A trajectory $c$ on the plane $p$ starting at $v$ and ending at $w$; (b), (c) and (d): Configurations illustrating Lemma 1.

**Proof.** Suppose first that the projection $v_1'$ of $v_1$ on to $p_2$ is neither null nor parallel to $v_2$, and that an arbitrary trajectory $c_1 : [0, \epsilon_1] \to \mathbb{S}^2$, starting at $v_1$ and lying on $p_1$, is given. Locate the startpoints of both $v_1$ and $v_2$ at a point $O$ on $p_1 \cap p_2$. See Figure 4(b). Define a trajectory $c_2 : [0, \epsilon_2] \to \mathbb{S}^2$, lying on $p_2$ and starting at $v_2$ and ending along $v_1'$, such that the angle between $c_2(t)$ and $v_1'$ decreases strictly and monotonically to 0 as $t$ increases from 0 to $\epsilon_2$. Such a trajectory $c_2$ is most easily defined by monotonically rotating a unit vector $v$ on $p_2$, initially identical with $v_2$, towards $v_1'$. It easily verified that the angle between $c_2(t)$ and $v_1$ decreases strictly and monotonically as well. By continuity considerations it may be seen that, if the speed of $c_2$ is set (by linearly scaling the parameter $t$) high enough with respect to that of $c_1$, then, indeed, there exists some positive $\delta \leq \min \epsilon_i$, such that the angle between $c_1(t)$ and $c_2(t)$ is strictly less than $\theta$ for $0 < t \leq \delta$. This proves the second claim of the lemma, as well as the first claim in the case that the projection $v_1'$ of $v_1$ on to $p_2$ is neither null nor parallel to $v_2$.

Suppose next that the projection $v_1'$ of $v_1$ on to $p_2$ is not null but is parallel to $v_2$. See Figure 4(c). Choose a non-null vector $w$ with startpoint at $O$ and lying along the line $p_1 \cap p_2$ such that the angle of $v_1'$ with $w$ is at most $\pi/2$. Define a trajectory $c_1 : [0, \epsilon_1] \to \mathbb{S}^2$ by monotonically rotating a unit vector $v$ on $p_1$, initially identical with $v_1$, towards $w$. Similarly, define a trajectory $c_2 : [0, \epsilon_2] \to \mathbb{S}^2$ by monotonically rotating a unit vector $v$ on $p_2$, initially identical with $v_2$, towards $w$. The trajectories are the dotted arrows in Figure 4(c). Then, $c_1$ and $c_2$ satisfy the requirements of the first claim of the lemma.

Finally, suppose that the projection $v_1'$ of $v_1$ on to $p_2$ is null, in other words, that $v_1$ is perpendicular to $p_2$. See Figure 4(d). Choose a non-null vector $w$ with startpoint at $O$ and lying along the line $p_1 \cap p_2$ such that the angle of $v_2$ with $w$ is at most $\pi/2$. Repeat the definition of the trajectories $c_1$ and $c_2$ in the previous case.

*Remark*: The second claim of the lemma cannot be made if the projection $v_1'$ of $v_1$ on to $p_2$ is either null or parallel to $v_2$. Say $v_1'$ is not null but is parallel to $v_2$. Then $v_2$ is angularly as close as it can be to $v_1$ while remaining on the plane $p_2$; now, if $c_1$ goes the "wrong" way from $v_1$ (e.g., if $c_1$ went in the direction opposite to that indicated in Figure 4(c)), then no trajectory $c_2$ can be defined to satisfy the second claim. A similar counter-example may be constructed if the projection $v_1'$ is null, i.e., if $v_1$ is perpendicular to $p_2$. □

**Definitions.** Let $C = \{v_1, v_2, \ldots, v_n\}$ be a configuration on a finite set of planes $F$. Construct a *complete undirected graph* $G_C$ on $n$ vertices *named* $g_1$, $g_2$, …, $g_n$, with the weight on the edge joining vertices $g_i$ and $g_j$ being the angle between the vectors $v_i$ and $v_j$. Call $G_C$ a *configuration graph*, in particular, *the graph of* $C$. Call edges of maximal weight in $G_C$ *maximal edges*. Denote by $G'_C$ the subgraph of $G_C$ consisting exactly of the maximal edges and their endvertices, call it the *subgraph of maximal edges*. See Figures 2(b) and (d). *Remark*: We shall often refer to vertices of $G_C$ as if *labeled* by vectors of $C$ – the reason we do not use the vectors themselves to name vertices is that there may be duplicates in a configuration.

If $C$ is an extreme configuration on $F$, obviously the weight of a maximal edge of $G_C$ is equal to $minmax(F)$, and the weights of other edges strictly less.

If $C = \{v_1, v_2, \dots, v_n\}$ is a configuration on a finite set of planes $F$ and two of the vectors $v_i$ and $v_j$, where $1 \leq i < j \leq n$, in $C$ and their corresponding planes $p_i$ and $p_j$ are such that the either

(a) the projection of $v_i$ on to $p_j$ is null or parallel to $v_j$, or

(b) the projection of $v_j$ on to $p_i$ is null or parallel to $v_i$,

then $v_i$ and $v_j$ are said to form a *projective pair* of $C$.

**Lemma 2** *If $F$ is a finite set of at least 3 planes and $C$ is an extreme configuration on $F$, containing at most one projective pair, then the subgraph of maximal edges $G'_C$ contains at least one (simple) cycle; in other words, $G_C$ contains a cycle of maximal edges.*

**Proof.** If $minmax(F) = 0$, the lemma hold trivially as all edges of $G_C$, for an extreme configuration $C$, have weight 0.

Assume then that $minmax(F) > 0$. We claim that, if $G'_C$ is acyclic, then $C$ cannot be extreme. In fact, we shall prove the following:

**Sublemma** Let $F = \{p_1, \dots, p_n\}$ and $minmax(F) > 0$. Let $C = \{v_1, \dots, v_n\}$, where each vector $v_i$ is of unit length and lies on $p_i$, $1 \leq i \leq n$, be a configuration on $F$ such that $G'_C$ is acyclic. Further, suppose $c$ contains at most one projective pair. Suppose, by renaming, if necessary, that the set of vertices of $G'_C$ is $\{g_1, \dots, g_m\}$, $m \leq n$.

Then there exist trajectories $c_i : [0, \epsilon_i] \rightarrow \mathbb{S}^2$, starting at $v_i$ and lying on $p_i$, for $1 \leq i \leq m$, such the $max(C(t)) < max(C)$, for $0 < t \leq \delta$, for some positive $\delta \leq \min \epsilon_i$, where the configuration $C(t) = \{c_1(t), \dots, c_m(t), v_{m+1}, \dots, v_n\}$.

*In the following and elsewhere "rotation" is often used informally – the use may be formalized in terms of trajectories.*

**Proof.** Before we begin the formal proof here is an intuition: our strategy is to use the "slack" in the non-maximal edges of $G_C$ to perturb the vectors corresponding to the vertices in $G'_C$ and bring them angularly closer, or, equivalently, decrease the weights of the maximal edges of $G_C$. The first claim of Lemma 1 allows us to start with any pair, in particular, we choose a projective pair of vectors corresponding to vertices in $G'_C$, if such a pair exists. Subsequently, the perturbation "ripples" to all vectors corresponding to vertices in $G'_C$ by repeated application of the second claim of Lemma 1 – acyclicity of $G'_C$ guarantees that there is no circular "deadlock".

The formal proof is by induction on the number $m$ of vertices in $G'_C$, which, of course, is at least 2. If $m = 2$, use the first claim of Lemma 1 to begin the induction – observing that sufficiently small rotations of $v_1$ and $v_2$ will not change the weight of any initially non-maximal edge enough for that edge to become maximal. See for example Figures 2(c) and (d), where $v_1$ and $v_2$ may be rotated simultaneously on their respective planes to reduce the angle, call it $\alpha$ (originally $2\pi/3$), between them, but without letting $\alpha$ drop below any of the other five angle weights.

Assume, inductively, that the claim is true if the number of vertices in $G'_C$ is $m \leq M$, for some given $M \geq 2$. Suppose now that $C$ is a configuration on $F$ so that $G'_C = \{g_1, \ldots, g_{M+1}\}$ has $M+1$ vertices. Consider a leaf vertex, say (renumbering if necessary) $g_{M+1}$ of $G'_C$ – in particular, $g_{M+1}$ is of degree 1 in $G'_C$ – and, again renumbering if necessary, assume that the unique vertex in $G'_C$ adjacent to $g_{M+1}$ is $g_M$. As $C$ contains at most one projective pair we can assume, w.l.o.g., that the pair $v_M$ and $v_{M+1}$ is not projective (see remark below).

See Figure 2(e) (in particular, vertex $g_5$). First, apply the inductive hypothesis to the configuration $C - \{v_{M+1}\}$ on the set of planes $F - \{p_{M+1}\}$ to obtain vector trajectories $c_i : [0, \epsilon_i] \to \mathbb{S}^2$, for $1 \leq i \leq M$, satisfying the claim of the sublemma.

Next, applying the second claim of Lemma 1 to the pair $v_M$ and $v_{M+1}$ (which is not projective), one can find a trajectory $c_{M+1} : [0, \epsilon_{M+1}] \to \mathbb{S}^2$, starting at $v_{M+1}$ and lying on $p_{M+1}$, so that the family of trajectories $c_i : [0, \epsilon_i] \to \mathbb{S}^2$, for $1 \leq i \leq M+1$, satisfy the claim of the sublemma – observing again that a sufficiently small rotation of $v_{M+1}$ will not change the weight of any initially non-maximal edge adjacent to $v_{M+1}$ enough for that edge to become maximal. □

**Lemma 3** *Let $F$ be a finite set of planes and $C$ a configuration of unit vectors on $F$ such that $g_{i_1}, \ldots, g_{i_m}, g_{i_{m+1}} = g_{i_1}$ is a simple cycle of length four or more in $G'_C$. Then, of the $m$ vectors $v_{i_1}, \ldots, v_{i_m}$, at most three are distinct; in other words, if $G_C$ contains a simple cycle of maximal edges, at most three of the vectors (precisely, vector labels) on that cycle are distinct.*

**Proof.** Suppose first that $G'_C$ contains a simple 4-cycle $g_{i_1}, g_{i_2}, g_{i_3}, g_{i_4}, g_{i_1}$ such that all four vectors $v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}$ are distinct. Parallel-translate the vectors $v_{i_k}$, $1 \leq k \leq 4$, so that their startpoints are all located at one point $O$. As the angles between successive pairs are equal, the endpoints of $v_{i_k}$, call them $p_{i_k}$, $1 \leq k \leq 4$, form the vertices of a spherical rhombus on a unit sphere centered at $O$, i.e., the great circle arcs joining $p_{i_1}p_{i_2}, p_{i_2}p_{i_3}, p_{i_3}p_{i_4}, p_{i_4}p_{i_1}$ are all of equal length. See Figure 5. It may be verified that, in a spherical rhombus, at least one of the two great circle diagonals joining opposite vertices is of greater length than the arc length of a side. This leads to the conclusion that one of the angles between either $v_{i_1}$ and $v_{i_3}$ or $v_{i_2}$ and $v_{i_4}$ is greater than the angle between a successive pair of $v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}, v_{i_1}$, contradicting that $g_{i_1}, g_{i_2}, g_{i_3}, g_{i_4}, g_{i_1}$ is a cycle *in* $G'_C$ (with all edge weights maximal in $G_C$). It follows that the initial hypothesis that all four of $v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}$ are distinct cannot hold. Cases of cycles of length greater than 4 may be argued similarly.

See Figure 2(b) for an example of a 4-cycle $g_1, g_2, g_3, g_4, g_1$ of maximal edges where only three of the vectors are distinct. □

## 3 Regular Polyhedra

If $F$ is the set of planes containing the faces of a polyhedron $P$, then a configuration on $F$ is simply called a configuration on $P$. Accordingly, define $minmax(P) = minmax(F)$. Henceforth, if $f$ denotes a face of a polyhedron, it shall also denote the plane containing that face – ambiguity will not be a problem.

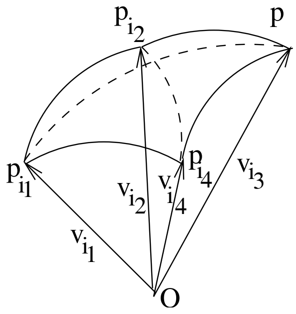


Figure 5: A spherical rhombus at the endpoints of a cycle of maximal edges.

We wish to determine extreme configurations (*modulo* rigid motions of $\mathbb{R}^3$) and the minmax angle for *regular polyhedra*. Clearly, these depend only on the type of the given regular polyhedron, i.e., its number of faces, and not its size.

The *cube* is simplest:

**Theorem 1** *The minmax angle of a cube is* $\pi/3$.

**Proof.** Observe first that the case of the cube reduces to finding extreme configurations on the set of three co-ordinate planes: as the (axis-parallel) cube has two faces parallel to each co-ordinate plane, one need only duplicate each vector from an extreme configuration on the co-ordinate planes.

Say $C = \{v_1, v_2, v_3\}$ is an extreme configuration on the co-ordinate planes with $v_1$, $v_2$ and $v_3$ lying on the $yz$-, $xz$-, and $xy$- coordinate planes. It may be verified easily that, if any two of the three vectors $v_1$, $v_2$ and $v_3$ form a projective pair, then one of these vectors must lie along a co-ordinate axis. Further, if one of the vectors $v_1$, $v_2$ and $v_3$ does indeed lie along a co-ordinate axis, it is seen that $max(C) \geq \pi/2$. However, configurations $C'$ on the co-ordinate planes, with $max(C') < \pi/2$, are trivially constructed "by hand".

It follows that the extreme configuration $C$ does not contain a single projective pair. One can, therefore, apply Lemma 2 to deduce that the angle between each pair from $v_1$, $v_2$ and $v_3$ is equal. It follows from the symmetry of the three coordinate planes that, in such a configuration, the angles $\alpha_1$, $\alpha_2$ and $\alpha_3$ made by $v_1$, $v_2$ and $v_3$ with the $y$-, $z$- and $x$-axis are equal as well. See Figure 6.

Consider the the isosceles triangle $OBC$ with unit sides lying along $v_2$ and $v_3$: the square of the third side is seen to be

$$BC^2 = \cos^2\alpha_2 + \sin^2\alpha_3 + (\cos\alpha_3 - \sin\alpha_2)^2 = 2(1 - \sin\alpha_2\cos\alpha_3) = 2(1 - \sin\alpha_2\cos\alpha_2),$$

using $\alpha_2 = \alpha_3$. This quantity is a minimum – therefore, the angle between $v_2$ and $v_3$ is a minimum – when $\alpha_2 = \pi/4$. When $\alpha_1 = \alpha_2 = \alpha_3 = \pi/4$, it is checked that, indeed, $max(C) = \angle BOC = \pi/3$. $\square$

Next, consider a *regular tetrahedron*. In this case, a "symmetric" disposition of an extreme configuration, as on a cube, is elusive.[1] In fact, results are far less trivial and somewhat curious.

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[1] It is amusing (and frustrating) to try, as we did, to discover an extreme configuration on a cardboard model of a regular tetrahedron with rotating arrows pinned to each face!

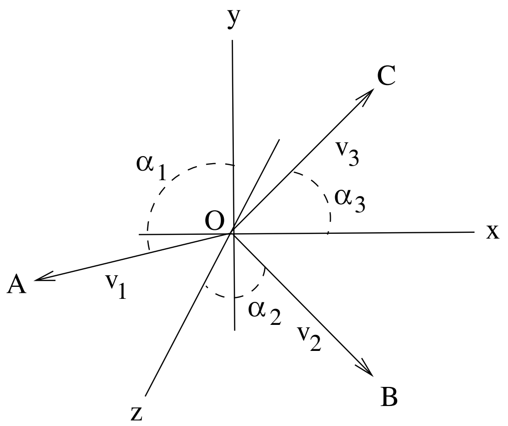


Figure 6: Computing an extreme configuration on the three co-ordinate planes.

We begin with a few definitions:

**Definitions.** Fix a regular tetrahedron $T$ and choose a set $F = \{f_1, f_2, f_3\}$ of 3 faces of $T$ meeting at a vertex $O$, so that the order $f_1, f_2, f_3$ is counter-clockwise around $O$, viewed from above $O$ – imagine $T$ placed with the face opposite $O$ as its base. Label the face opposite $O$ as $f_4$.

Choose a direction, call it the *standard axis*, on each of $f_1, f_2, f_3$ in the following symmetric manner: the standard axis on $f_i$ is along the edge opposite $O$ in $f_i$, oriented by a counter-clockwise, viewed from above, traversal of the perimeter of $f_4$. Label the vertices of $f_4$ as $A, B, C$, where the edges $AB$, $BC$, and $CA$ lie on faces $f_1$, $f_2$, and $f_3$ and each is oriented along the standard axis. See Figure 7(a).

Now that a standard axis has been chosen on each face of $F$, given any direction, i.e., a non-null vector, $v$ on a face $f_i \in F$, define *the angle of* $v$ on face $f_i$ to be $\theta$, $0 \leq \theta < 2\pi$, the angle measured counter-clockwise, viewed from above, from the standard axis on $f_i$ to $v$.

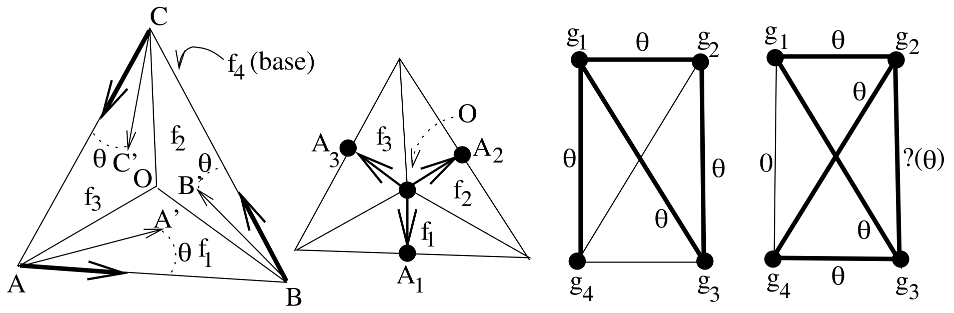
An *equicycle* of vectors on $F$ is a configuration $C = \{v_1,\ v_2,\ v_3\}$ on $F$ such that the *angle between each pair of vectors* from $C$ is equal to the same value $\phi$, which is called the *angle of the equicycle*.

A *regular equicycle* of vectors on $F$ is an equicycle so that the *angle of each vector on its corresponding face* is the same. See Figure 7(a). *Remark*: It may appear somewhat unintuitive that there exist irregular equicycles on $F$, which is probably why the following crucial lemma has a surprisingly non-trivial proof – keep the face labeling scheme above in mind:

**Lemma 4** *Let $A_1, A_2, A_3$ be the mid-points of the sides opposite $O$ in the faces $f_1, f_2, f_3$. Then extreme configurations of $F = \{f_1, f_2, f_3\}$ are precisely the configurations of the form $\alpha_1 OA_1, \alpha_2 OA_2, \alpha_3 OA_3$ where $\alpha_1, \alpha_2, \alpha_3$ are arbitrary non-zero scalars, either* all positive *or* all negative. *See Figure 7(b).*

*Further, any configuration of vectors $\{v_1, v_2, v_3\}$ on $F$ that is not extreme can be perturbed an arbitrarily small amount to a configuration with smaller maximal angle. In other words, the function $max(C)$ on the space of configurations $C$ on $F$ has no local minimum that is not a global minimum.*

*Remark*: The first claim of Lemma 4 is geometrically intuitive: an extreme configuration



(a) (b) (c) (d)

Figure 7: (a) Standard axes (bold) and a regular equicycle $\{AA', BB', CC'\}$ on $F$; (b) An extreme configuration on $F$; (c) and (d): Graphs of configurations for Lemma 5 – subgraph of maximal edges is bold.

of $F$ is situated symmetrically on the faces $f_i$, $1 \leq i \leq 3$, looking out from $O$.

We defer the somewhat lengthy proof of Lemma 4 till after our main theorem. But first we need another preliminary lemma.

**Lemma 5** *If $C$ is an extreme configuration of unit vectors on a regular tetrahedron $T$, then $G'_C$ contains a simple 4-cycle; in other words, $G_C$ contains a simple 4-cycle of maximal edges.*

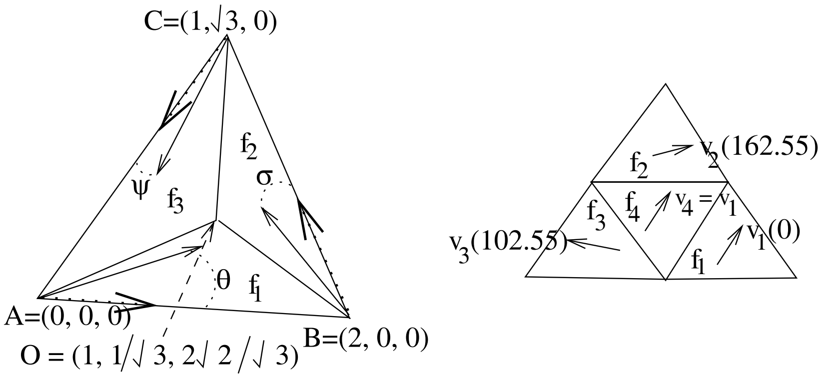
*Further, relabeling faces of $T$ if necessary, $C$ is of the form $\{v_1, v_2, v_3, v_4\}$, $v_i$ lying on face $f_i$, $1 \leq i \leq 4$, of $T$, where (a) $v_1 = v_4$ lies on the edge shared by $f_1$ and $f_4$, and (b) $v_1, v_2, v_3$ label vertices of a cycle of maximal edges in $G_C$.*

**Proof.** For the first statement of the lemma, let $C = \{v_1, v_2, v_3, v_4\}$, with $v_i$ lying on face $f_i$ of $T$, $1 \leq i \leq 4$. We intend to apply Lemma 2. But first, as for Theorem 1, we must eliminate the possibility that $C$ contains more than one projective pair: in fact, verifying that a configuration on a regular tetrahedron that contains more than one projective pair cannot be extreme is elementary but tedious, and we omit details. Therefore, as $C$ is an extreme configuration, we apply Lemma 2 to deduce that $G_C$ contains at least a simple 3-cycle of maximal edges. Suppose, if possible, that it does not contain a simple 4-cycle of maximal edges. Say then, w.l.o.g., that $g_1, g_2, g_3$ are the vertices in a simple 3-cycle of maximal edges of $G_C$ corresponding – recall the definition of $G_C$ – to the vectors $v_1$, $v_2$, $v_3$.

Denote by $\theta$ the minmax angle of a regular tetrahedron, i.e., the weight of a maximal edge in $G_C$. By rotating $v_4$ on $f_4$, if necessary, one can assume that it makes an angle of $\theta$ with at least one of $v_1, v_2, v_3$ – w.l.o.g., assume that the angle between $v_4$ and $v_1$ is $\theta$. Now, both the angle between $v_4$ and $v_2$ and the angle between $v_4$ and $v_3$ must be less than $\theta$, for, if not, we would have a simple 4-cycle of maximal edges. Therefore, Figure 7(c) is a corresponding depiction of $G_C$.

Consider the set of planes $H = \{f_2, f_3, f_4\}$. By Lemma 1, there is a perturbation of $v_2$ and $v_3$ on the faces $f_2$ and $f_3$, small enough that it reduces the angle between $v_2$ and $v_3$, but does not increase either the angle between $v_4$ and $v_2$ or the angle between $v_4$ and $v_3$ to the value $\theta$. One, therefore, reaches a configuration on $H$ whose maximal angle is less than $\theta$. This implies that $minmax(H) < \theta$.

Next, consider the set of planes $F = \{f_1, f_2, f_3\}$. There is a rigid motion of $\mathbb{R}^3$ that takes $F$ to $H$, i.e., maps the 3 planes of $F$ one-one to those of $H$. It follows that $minmax(F) < \theta$



(a) (b)

Figure 8: (a) Regular tetrahedron of side length 2 – standard axes on $f_1, f_2, f_3$ bold; (b) An extreme configuration on a regular tetrahedron with the angle in degrees of vectors on faces $f_1$, $f_2$ and $f_3$ (see definitions preceding Lemma 4) indicated.

as well. Therefore, the configuration $D = \{v_1, v_2, v_3\}$ on $F$ is not extreme.

As the configuration $D$ on $F$ is not extreme, it can be perturbed, using the second claim of Lemma 4, an arbitrarily small amount to a configuration $D'$, such that $max(D') < \theta$. Since the angles between $v_4$ and $v_2$ and $v_4$ and $v_3$ are both less than $\theta$, one can find a perturbation (mimicking exactly the argument in the last inductive step of Lemma 2) of the configuration $C = \{v_1, v_2, v_3, v_4\}$ to a configuration $C'$, so that $max(C') < \theta$: roughly, as $v_1$, $v_2$ and $v_3$ rotate towards the configuration $D'$, also rotate $v_4$ towards $v_1$ reducing the angle between them, at the same time using the "slack" in the angles between $v_4$ and $v_2$ and $v_4$ and $v_3$ to not let these angles grow to $\theta$. This, of course, contradicts that $C$ is an extreme configuration.

We conclude that $G_C$ indeed contains a simple 4-cycle of maximal edges. This proves the first statement of the lemma.

For the second statement, observe first that, by Lemma 3, at most three of the $v_i$ are distinct. Clearly, not all the $v_i$ are identical, given that planes of a tetrahedron do not intersect in one straight line. If only two of the $v_i$ are distinct, say, w.l.o.g., $v_1$ and $v_2$, then each must label a pair of vertices of $G_C$ and, therefore, each must lie on the edge shared by the corresponding pair of faces of $T$. We conclude that, in such a case, $v_1$ and $v_2$ would lie one each on two skew (non-intersecting) edges of $T$. However, it is easily verified that no such configuration is extreme. We conclude that exactly three of the vectors $v_i$ are distinct.

Relabeling faces of $T$ if necessary, assume that $v_1 = v_4$ and that $\{v_1, v_2, v_3\}$ is a set of distinct vectors. Since $v_1$ and $v_4$ lie on the faces $f_1$ and $f_4$, it follows that $v_1 = v_4$ lies on the edge shared by $f_1$ and $f_4$. Therefore, in this case, a simple 4-cycle in $G'_C$ must be $g_1, g_2, g_4, g_3, g_1$. Accordingly, the graph $G_C$ is as depicted in Figure 7(d), where one must show that the angle between $v_2$ and $v_3$ is indeed $\theta$ and not less. However, if the angle between $v_2$ and $v_3$ were less than $\theta$, then $v_2$ and $v_3$ could be rotated simultaneously towards $v_1 (= v_4)$ to reach a configuration whose maximal angle is less than $\theta$, contradicting the hypothesis on $\theta$. This proves the second statement. □

With Lemma 5 in hand we present our main theorem, whose proof now reduces to solving a "finite and local" optimization problem.

**Theorem 2** *The minmax angle of a regular tetrahedron is* $\cos^{-1}\frac{-1+\sqrt{17}}{8} \approx 67.0206°$.

**Proof.** By Lemma 5, an extreme configuration $C$ on a regular tetrahedron is of the form $\{v_1, v_2, v_3, v_4\}$, where (a) $v_i$ lies on a face $f_i$, $1 \leq i \leq 4$, of $T$, (b) $v_1 = v_4$ lie on the edge shared by $f_1$ and $f_4$ (and may be assumed oriented along the standard axis of $f_1$), and (c) $v_1, v_2, v_3$ are the vertices of a cycle of maximal edges in $G_C$. Consider the regular tetrahedron of side length 2 as located in Figure 8(a): $A$, $B$, $C$ and $O$ are the points $(0,0,0)$, $(2,0,0)$, $(1,\sqrt{3},0)$ and $(1,1/\sqrt{3},2\sqrt{2}/\sqrt{3})$, and $f_1, f_2, f_3$ are the faces $OAB, OBC, OCA$. We, therefore, need now determine all equicycles $\{v_1, v_2, v_3\}$ of vectors on $F = \{f_1, f_2, f_3\}$, such that $v_1$ is directed along the standard axis on $f_1$: so, if $v_1, v_2, v_3$ are of angles $\theta, \sigma, \psi$ on $f_1, f_2, f_3$, then $\theta = 0$. Assuming all $v_i$, $1 \leq i \leq 4$, to be of unit length, we have from trigonometry on Figure 8(a):

$$v_1 \;= i \tag{1}$$

$$v_2 = (-\frac{1}{2}\cos\sigma - \frac{1}{2\sqrt{3}}\sin\sigma)\,i + (\frac{\sqrt{3}}{2}\cos\sigma - \frac{1}{6}\sin\sigma)\,j + \frac{2\sqrt{2}}{3}\sin\sigma\,k \tag{2}$$

$$v_3 = (-\frac{1}{2}\cos\psi + \frac{1}{2\sqrt{3}}\sin\psi)\,i + (-\frac{\sqrt{3}}{2}\cos\psi - \frac{1}{6}\sin\psi)\,j + \frac{2\sqrt{2}}{3}\sin\psi\,k \tag{3}$$

As $\angle v_1v_2 = \angle v_1v_3$ one has $v_1 \cdot v_2 = v_1 \cdot v_3$, so that

$$(-\frac{1}{2}\cos\sigma - \frac{1}{2\sqrt{3}}\sin\sigma) = (-\frac{1}{2}\cos\psi + \frac{1}{2\sqrt{3}}\sin\psi) \tag{4}$$

which solves to

$$\psi = \sigma - \frac{\pi}{3} \quad \text{or} \quad \psi = -\sigma.$$

If $\psi = \sigma - \frac{\pi}{3}$ then one has from equation (3) that

$$v_3 = (-\frac{1}{2}\cos\sigma - \frac{1}{2\sqrt{3}}\sin\sigma)\,i + (-\frac{1}{2\sqrt{3}}\cos\sigma - \frac{5}{6}\sin\sigma)\,j + (\frac{\sqrt{2}}{3}\sin\sigma - \frac{\sqrt{2}}{\sqrt{3}}\cos\sigma)\,k \tag{5}$$

As $\angle v_2v_3 = \angle v_1v_2$ one has $v_2 \cdot v_3 = v_1 \cdot v_2$, and so from equations (1), (2) and (5) follows

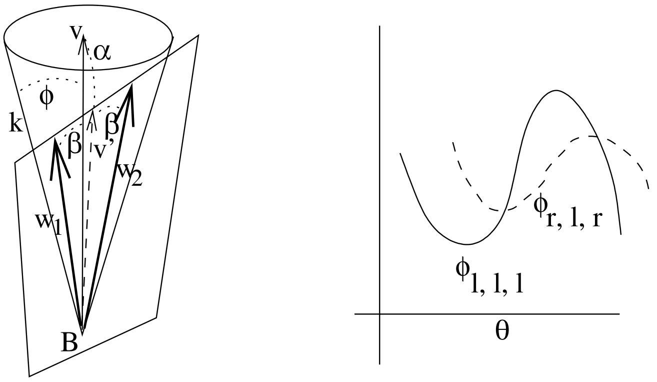
$$\frac{2}{3}\sin^2\sigma - \frac{2}{\sqrt{3}}\sin\sigma\cos\sigma \;=\; -\frac{1}{2}\cos\sigma - \frac{1}{2\sqrt{3}}\sin\sigma \tag{6}$$

Writing $X = v_1 \cdot v_2 = -\frac{1}{2}\cos\sigma - \frac{1}{2\sqrt{3}}\sin\sigma$, one has by manipulating equation (6) that $4X^2 + X - 1 = 0$, which solves to $X = \frac{-1\pm\sqrt{17}}{8}$, i.e., $X = 0.3904$ or $X = -0.6404$. The corresponding angles of the equicycles are $\cos^{-1} X \approx 67.0206°$ or $129.8217°$.

If $\psi = -\sigma$ then, calculating similarly, $X = v_1 \cdot v_2$ satisfies the quadratic $2X^2 - X - 1 = 0$, which solves to $X = 1$ or $X = -\frac{1}{2}$. $X = -\frac{1}{2}$ corresponds to a regular equicycle of angle $120°$, where each of $v_1$, $v_2$, $v_3$ lies on and is oriented either along or opposite the standard axis of its face; $X = 1$ does not give a feasible solution.

The theorem follows. See Figure 8(b) for an extreme configuration of a regular tetrahedron depicted on an unfolding. □

**Proof of Lemma 4.** We intend to apply Lemma 2: again (as for Lemma 5), it may be checked "by hand" that configurations on F with more than one projective pair cannot be extreme. Therefore, by Lemma 2, to determine extreme configurations on $F$ one need



(a) (b)

Figure 9: (a) Cone $k$ with semi-vertical angle $\phi$ for the proof of Lemma 4; (b) Hypothetical graphs of $\phi_{\text{left, left, left}}$ and $\phi_{\text{right, left, right}}$.

consider only equicycles. Consequently, let us examine how equicycles are formed on $F$. Consider a unit vector $v$ on $f_1$ of angle $\theta$ (see again definitions preceding the statement of the lemma and Figure 7(a) for the labeling scheme corresponding to $F$). Locate the startpoint of $v$ at $B$. Let the projection of $v$ on to $f_2$ be the vector $v'$ of angle $\gamma = f(\theta)$ on $f_2$ – the function $f$ is well-defined as the projection on to $f_2$ of any non-null vector on $f_1$ is again non-null. Let the angle between $v$ and $v'$ be $\alpha = h(\theta)$: $h$ is well-defined such that always $0 \leq h(\theta) < \pi/2$.

Consider now a (possibly degenerate) cone $k$ with semi-vertical angle $\phi$, $0 \leq \phi \leq \pi$, apex at $B$, and axis along $v$ (if $\phi > \pi/2$, imagine a cone with semi-vertical angle $\pi - \phi$ and axis along $-v$). There are three possibilities:

1. $\phi < h(\theta)$ or $\phi > \pi - h(\theta)$, when $k$ does not intersect $f_2$ except at $B$: there is no vector on $f_2$ making an angle $\phi$ with $v$.

2. $\phi = h(\theta)$ or $\phi = \pi - h(\theta)$, when $k$ is tangential to $f_2$: there is exactly one *unit* vector (the length of the vector is immaterial but we fix it to avoid clumsy statements) on $f_2$ making an angle $\phi$ with $v$.

3. $h(\theta) < \phi < \pi - h(\theta)$, when $k$ intersects $f_2$ in two distinct straight lines: there are exactly two unit vectors on $f_2$ making an angle $\phi$ with $v$.

It is seen by symmetry, in cases 2 and 3, that the two (possibly equal) unit vectors $w_1$ and $w_2$ on $f_2$, making an angle of $\phi$ with $v$, are at angles $f(\theta) \pm \beta$ on $f_2$ (see Figure 9(a)), where again $\beta = g(\theta, \phi)$ is a well-defined function with range $0 \leq g(\theta, \phi) \leq \pi$ in the domain where $h(\theta) \leq \phi \leq \pi - h(\theta)$. Observe that $g(\theta, \phi) = 0$ if $\phi = h(\theta)$, and $g(\theta, \phi) = \pi$ if $\phi = \pi - h(\theta)$. Distinguish, if required, between $w_1$ and $w_2$ by declaring that $w_i$ is the *right* (*left*) *vector* on $f_2$ making an angle of $\phi$ with $v$, if $w_i$ lies in the half-space in $f_2$ of the straight line containing $v'$ that is to the right (left) of the observer who is standing on $f_2$ to the outside of $T$ and facing in the direction of $v'$. Given our standard axes and angle measuring conventions, the right vector on $f_2$ making an angle of $\phi$ with $v$ is of angle $f(\theta) - g(\theta, \phi)$ on $f_2$; similarly, the left vector on $f_2$ making an angle of $\phi$ with $v$ is of angle $f(\theta) + g(\theta, \phi)$ on $f_2$.

Next, we describe a procedure to derive all equicycles of unit vectors containing the particular vector $v$ of given angle $\theta$ on $f_1$. Fix a *governing tuple* $(X_1, X_2, X_3)$ where each $X_i$, $1 \leq i \leq 3$, has the value "left" or "right". Solve – though we shall not explicitly do so – the following 3 equations in 3 unknowns to find $\phi$, $\theta'$ and $\theta''$:

$$\begin{aligned} \theta' &= f(\theta) \pm g(\theta, \phi) && (7) \\ \theta'' &= f(\theta') \pm g(\theta', \phi) && (8) \\ \theta &= f(\theta'') \pm g(\theta'', \phi), && (9) \end{aligned}$$

where the sign on the RHS of equation $(i+6)$, $1 \leq i \leq 3$, is $-$ or $+$ according as $X_i$ is "right" or "left". Intuitively, start with $v$ on $f_1$, choose one of the two vectors on $f_2$ at angle of $\phi$ to $v$, choose again one of two vectors on $f_3$ at angle $\phi$ to the one chosen on $f_2$, finally choose a vector on $f_1$ at angle $\phi$ to the one chosen on $f_3$, and this last vector must be $v$ again (by the symmetry of the tetrahedron the same functions $f$ and $g$ apply at each step).

The at most 8 equicycles of unit vectors containing the vector $v$ are obtained by solving the 8 sets of equations as above, corresponding to the 8 choices of the governing tuple $(X_1, X_2, X_3)$, and, accordingly, determining the angle $(\phi)$ of the equicycle and the angles $(\theta'$ and $\theta'')$ of the other two vectors in the equicycle on their respective faces. In other word, $\phi$, $\theta'$ and $\theta''$ are determined as functions of $\theta$, for each choice of the governing tuple $(X_1, X_2, X_3)$. Denote by $\phi_{X_1,X_2,X_3}$ the function $\phi$ of $\theta$ for the choice $(X_1, X_2, X_3)$.

Given our standard axes and angle measuring conventions it may be seen that a (left, left, left) choice for the governing tuple gives exactly all regular equicycles.

Clearly, we need to determine, for each choice of the governing tuple $(X_1, X_2, X_3)$, the minima of $\phi_{X_1,X_2,X_3}$ as a function of $\theta$. Attempting this directly by computing an explicit expression of $\phi_{X_1,X_2,X_3}$ in terms of $\theta$ and then differentiating to use calculus seems forbiddingly complicated. Instead, we apply a 3-step plan based on determining certain properties of the functions involved:

(a) Observing that a (left, left, left) choice of the governing tuple gives regular equicycles, find the maxima and minima in this case by elementary means.

(b) Analyze the simultaneous equations in $\theta$, $\theta'$, $\theta''$ and $\phi$, given $\frac{d\phi_{X_1,X_2,X_3}}{d\theta} = 0$, to conclude that the *number of values* of $\theta$ at which $\phi_{X_1,X_2,X_3}$ is extreme is equal, for all choices of the governing tuple $(X_1, X_2, X_3)$.

(c) Use the above two to draw conclusions for the cases of the irregular equicycles.

(a) Locate a regular tetrahedron of side length 2 as in Figure 8(a), where faces $f_1, f_2, f_3$ are $OAB, OBC, OCA$. Consider a *regular* equicycle of unit length vectors $v_1, v_2, v_3$ lying on $f_1, f_2, f_3$ of angle $\theta$ in their respective faces (so that $\theta = \sigma = \psi$ in Figure 8(a)). Then, by trigonometry:

$$\begin{aligned} v_1 &= \cos\theta\, i + \frac{1}{3}\sin\theta\, j + \frac{2\sqrt{2}}{3}\sin\theta\, k && (10) \\ v_2 &= (-\frac{1}{2}\cos\theta - \frac{1}{2\sqrt{3}}\sin\theta)\, i + (\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{6}\sin\theta)\, j + \frac{2\sqrt{2}}{3}\sin\theta\, k && (11) \\ \text{(therefore)}\quad v_1 \cdot v_2 &= -\frac{1}{2}\cos^2\theta + \frac{5}{6}\sin^2\theta && (12) \end{aligned}$$

Differentiating equation (12) w.r.t. $\theta$, one sees that $\phi_{\text{left, left, left}} = \cos^{-1}(v_1 \cdot v_2)$ has two (global) maxima at $\theta = 0, \pi$, and two (global) minima at $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

One concludes that (i) on the space of regular equicycles $C$, $max(C)$ is a global minimum exactly at the configurations described in the first part of the lemma, and (ii) a regular equicycle $C$, where $max(C)$ is not a global minimum, is not a local minimum either, i.e., it can be perturbed an arbitrarily small amount to a regular equicycle with smaller maximal angle.

(b) From equations (7) and (8) above, for a given choice of the governing tuple $(X_1, X_2, X_3)$, one has the following two equations (the ± sign depending on $X_1$ and $X_2$):

$$\begin{aligned} d\theta' &= \frac{df}{d\theta} d\theta' \pm \frac{\partial g}{\partial \theta} d\theta \pm \frac{\partial g}{\partial \phi_{X_1,X_2,X_3}} d\phi_{X_1,X_2,X_3} \qquad (13) \\ d\theta'' &= \frac{df}{d\theta'} d\theta' \pm \frac{\partial g}{\partial \theta'} d\theta' \pm \frac{\partial g}{\partial \phi_{X_1,X_2,X_3}} d\phi_{X_1,X_2,X_3} \qquad (14) \end{aligned}$$

From equation (9) one has that (the ± sign depending on $X_3$)

$$\begin{aligned} 0 &= d(f(\theta'') \pm g(\theta'', \phi_{X_1,X_2,X_3}) - \theta) \\ &= \frac{df}{d\theta''} d\theta'' \pm \frac{\partial g}{\partial \theta''} d\theta'' \pm \frac{\partial g}{\partial \phi_{X_1,X_2,X_3}} d\phi_{X_1,X_2,X_3} - d\theta \\ &= \ldots \text{ using equations (14) and (13) to substitute } d\theta'' \text{ and then } d\theta' \\ &= (\frac{df}{d\theta''} \pm \frac{\partial g}{\partial \theta''})(\frac{df}{d\theta'} \pm \frac{\partial g}{\partial \theta'})(\frac{df}{d\theta} \pm \frac{\partial g}{\partial \theta}) d\theta \\ &\quad \pm(\ (\frac{df}{d\theta''} \pm \frac{\partial g}{\partial \theta''})(\frac{df}{d\theta'} \pm \frac{\partial g}{\partial \theta'}) \pm (\frac{df}{d\theta''} \pm \frac{\partial g}{\partial \theta''} + 1)\ ) \frac{\partial g}{\partial \phi_{X_1,X_2,X_3}} d\phi_{X_1,X_2,X_3} - d\theta \qquad (15) \end{aligned}$$

Dividing equation (15) throughout by $d\theta$ and setting $\frac{d\phi_{X_1,X_2,X_3}}{d\theta} = 0$ one obtains

$$(\frac{df}{d\theta''} \pm \frac{\partial g}{\partial \theta''})(\frac{df}{d\theta'} \pm \frac{\partial g}{\partial \theta'})(\frac{df}{d\theta} \pm \frac{\partial g}{\partial \theta}) = 1 \qquad (16)$$

Equations (7), (8), (9) and (16) give 4 simultaneous equations in 4 unknowns that solve, in particular, to give values of $\theta$ when $\phi_{X_1,X_2,X_3}$ is an extreme value: observing that, for different choices of the governing tuple $(X_1, X_2, X_3)$, the summands in the 4 equations differ only in signs of their coefficients, one concludes that the number of values of $\theta$ at which $\phi_{X_1,X_2,X_3}$ is extreme is same for all choices of $(X_1, X_2, X_3)$.

(c) Since, by (a), there are 4 values of $\theta$ at which $\phi_{\text{left, left, left}}$ is extreme, it follows, from (b), that $\phi_{X_1,X_2,X_3}$ is extreme for 4 values of $\theta$, as well, for each $(X_1, X_2, X_3)$.

Now, for each $(X_1, X_2, X_3)$, it is seen that there are at least two maximum and two minimum values (both of which are global and so local as well) of $\phi_{X_1,X_2,X_3}$. In particular, by compactness and the non-constantness of $\phi_{X_1,X_2,X_3}$, there is at least one global maximum and one global minimum; reversing the vectors in the configurations corresponding to these two extrema gives another maximum and minimum. As the total of values of $\theta$ at which $\phi_{X_1,X_2,X_3}$ is extreme is known to be 4, one concludes that there are *exactly* 2 values of $\theta$ where $\phi_{X_1,X_2,X_3}$ is a maximum and 2 where it is a minimum, for any $(X_1, X_2, X_3)$.

Next, we claim that each of these maximum and minimum values, for *any* governing tuple $(X_1, X_2, X_3)$, corresponds, *also*, to a regular equicycle. For, if not, one would have a minimum value of some $\phi_{X_1,X_2,X_3}$ from an irregular equicycle of vectors $v_1, v_2, v_3$ of angles $\theta_1, \theta_2, \theta_3$ (so *not all equal* by irregularity) on the faces $f_1, f_2, f_3$. By the symmetry of a regular tetrahedron, one could then derive at least one more value of $\theta$, other than $\theta_1$, at which $\phi_{X_1,X_2,X_3}$ is a minimum, by cyclically permuting the vectors $v_1, v_2, v_3$ to configurations consisting of vectors of angles $(\theta_2, \theta_3, \theta_1)$ and $(\theta_3, \theta_1, \theta_2)$ on $(f_1, f_2, f_3)$. This would contradict the bound on the number of values of $\theta$ at which $\phi_{X_1,X_2,X_3}$ is minimum.

To complete the proof of the second part of the lemma, observe that, if $C$ is a configuration at which $max(C)$ is a local minimum in the space of configurations, then $C$ must be an equicycle arising from some choice of a governing tuple $(X_1, X_2, X_3)$, where $\phi_{X_1,X_2,X_3}$ is a local minimum value for the corresponding value of $\theta$ (the angle of the vector in $C$ on face $f_1$). From the arguments above, one sees that $C$ belongs to the space of regular equicycles, where we know that configurations where the maximal angle is minimum are precisely those described in the first part of the lemma; further, we also know that *any other* regular equicycle can be perturbed, in the space of regular equicycles, to reduce its maximal angle. Intuitively, the minima (and maxima) of $\phi_{X_1,X_2,X_3}$, for any $(X_1, X_2, X_3)$, occur where the graph of $\phi_{X_1,X_2,X_3}$ intersects the graph $\phi_{\text{left, left, left}}$ (see Figure 9(b)). $\Box$

Tabulated next are minmax angles of regular polyhedra that we have calculated to date (see notes following).

| Regular polyhedron (faces) | Minmax angle in approx. degrees |
|---|---|
| Tetrahedron (4) | 67.02 |
| Cube (6) | 60.00 |
| Octahedron (8) | 67.02 |
| Dodecahedron (12) | ? |
| Icosahedron (20) | ? |

1. The faces of an *octahedron* can be split into four parallel pairs. We, therefore, need only consider four mutually non-parallel faces; precisely, choose four faces adjacent to any one vertex. Results analogous to those for a tetrahedron can be proved. Surprisingly, the same quadratics arise for the equicycles on three (of the chosen four) faces of the octahedron as arise in the proof of Theorem 2 for a tetrahedron: we do not have a clear understanding at present, but suspect this phenomenon is due to the symmetric situation of two of the three faces w.r.t. the fixed vector on the third. Consequently the minmax angles of the tetrahedron and octahedron are identical.

2. To date our calculations for the *dodecahedron* and *icosahedron* are incomplete.

## 4 Conclusion & Open Problems

The complexity of computing joint separation is always of interest. Even in the case of planes in $\mathbb{R}^3$ that we have discussed, the question of how to efficiently determine the joint separation, i.e., minmax angle, of arbitrary finite sets remains open, though the general results from Section 2 could be useful. Of immediate interest, of course, is completing calculations of the joint separation of regular polyhedra, in particular, that of the dodecahedron and a icosahedron. Attempts that we have made to extend our methods to these two polyhedra lead to a profusion of different configuration graphs that seem to require individual computations. New insights leading to a more elegant approach would be welcome.

Another problem is that, in order to apply Lemma 2, we had to, in each case, prove "by hand" that configurations with more than one projective pair could not be extreme. This has been only tedium so far but will be significantly more difficult if there is a large number of planes. In fact, we conjecture that if $F$ is a finite set of at least three mutually non-parallel planes, such that $minmax(F) > 0$, then no configuration with even one projective pair can be extreme.

It would be useful, as well, to prove general theoretical results about joint separation independent of the setting. For example, a trivial result is $minmax(S') \leq minmax(S)$, if $S' \subset S$. More interesting results would likely require at least some assumptions on the *regular* subsets of $M$, i.e., subsets associated with some object of $\mathcal{S}$. Comparisons with known measures of separations of clusters [2, 4] should also be made.

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